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The Structure of a Class of Generalized Sasakian-Space-Forms

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Abstract: The object of the present paper is to study ξ-conformally flat and ϕ-conformally flat generalized Sasakian-space-forms.

Key words: ξ-conformally flat, ϕ-conformally flat, Sasakian manifold, Einstein manifold, η-Einstein manifold, conformally flat.

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1. Introduction

The nature of a Riemannian manifold mostly depends on the curvature tensor R of the manifold. It is well known that the sectional curvatures of a manifold determine curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as real space-form and its curvature tensor is given by

$$R(X,Y)Z = c\big\{g(Y,Z)X - g(X,Z)Y\big\}.$$

A Sasakian manifold with constant ϕ-sectional curvature is a Sasakian-space-form and it has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such space-forms in a common frame P. Alegre, D. E. Blair and A. Carriazo introduced the notion of generalized Sasakian-space-forms in 2004 [1]. In this connection it should be mentioned that in 1989 Z. Olszak [13] studied generalized complex-space-forms and proved its existence. A generalized Sasakian-space-forms are defined as follows:

Given an almost contact metric manifold $M(\phi,\xi,\eta,g)$, we say that M is a generalized Sasakian-space-form if there exist three fuctions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$
\begin{aligned}
R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} \\
+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}.
\end{aligned} \tag{1.1}
$$

for any vector fields X, Y, Z on M. In such a case we denote the manifold as $M(f_1,f_2,f_3)$. In [1] the authors cited several examples of generalized Sasakian-space-forms. If $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, then a generalized Sasakian-space-form with Sasakian structure becomes Sasakian-space-form. In [12], U. K. Kim studied conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms. He proved that some geometric properties of generalized Sasakian-space-form depend on the nature of the fuctions f_1, f_2 and f_3. Generalized Sasakian-space-forms have also been studied in the papers [2], [3], [4], [5], [9], [10], [12] and [13].

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g). At each point $p \in M$, decompose the tangent space T_pM into direct sum $T_pM = \phi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of T_pM generated by ξ_p. Thus the conformal curvature tensor C is a map

$$C : T_pM \times T_pM \times T_pM \longrightarrow \phi(T_pM) \oplus \{\xi_p\}, \quad p \in M.$$

It may be natural to consider the following particular cases:

(1) $C : T_p(M) \times T_p(M) \times T_p(M) \longrightarrow \{\xi_p\}$, i.e, the projection of the image of C in $\phi(T_p(M))$ is zero.

(2) $C : T_p(M) \times T_p(M) \times T_p(M) \longrightarrow \phi(T_p(M))$, i.e, the projection of the image of C in $\{\xi_p\}$ is zero. This condition is equivalent to

$$C(X,Y)\xi = 0. \tag{1.2}$$

(3) $C : \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \longrightarrow \{\xi_p\}$, i.e, when C is restricted to $\phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M))$, the projection of the image of C in $\phi(T_p(M))$ is zero. This condition is equivalent to

$$\phi^2 C(\phi X,\phi Y)\phi Z = 0. \tag{1.3}$$

A generalized Sasakian-space-form satisfying (1.2) and (1.3) is called ξ-conformally flat and ϕ-conformally flat respectively. A K-contact manifold satisfying the cases (1), (2) and (3) is considered in [14], [15] and [16] respectively.

Definition 1.1. A $(2n+1)$-dimensional generalized Sasakian-space-form is said to be ξ-conformally flat if

$$C(X,Y)\xi = 0, \tag{1.4}$$

where $X, Y \in T(M)$.

Definition 1.2. A $(2n+1)$-dimensional generalized Sasakian-space-form is said to be ϕ-conformally flat if

$$g\big(C(\phi X, \phi Y)\phi Z, \phi W\big) = 0. \tag{1.5}$$

In [15], it is proved that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstein Sasakian manifold. In [11], De and Biswas ξ-conformally flat $N(k)$-contact metric manifolds. A compact ϕ-conformally flat K-contact manifold with regular contact vector field has been studied in [16]. Moreover, in [6], Arslan, Murathan and Özgür studied ϕ-conformally flat (k,μ)-contact metric manifold. Motivated by the above studies, in this paper we study ξ-conformally flat and ϕ-conformally flat generalized Sasakian-space-forms.

The present paper is organized as follows:

After preliminaries in Section 3, we study ξ-conformally flat generalized Sasakian-space-forms and prove that a generalized Sasakian-space-form is always ξ-conformally flat. Section 4 deals with ϕ-conformally flat generalized Sasakian-space-forms and we prove that the space-form is ϕ-conformally flat if and only if $f_2 = 0$. As a consequence of the result we obtain some important corollaries.

2. PRELIMINARIES

In an almost contact metric manifold we have ([7], [8])

$$\phi^2(X) = -X + \eta(X)\xi, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0, \tag{2.4}$$

$$g(\phi X, \xi) = 0. \tag{2.5}$$

Again for a $(2n+1)$-dimensional generalized Sasakian-space-form we have ([1])

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.6}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.7}$$

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.8}$$

$$R(\xi, X)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X)], \tag{2.9}$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{2.10}$$

$$S(\xi, \xi) = 2n(f_1 - f_3), \tag{2.11}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.12}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.13}$$

where R, S and r are the curvature tensor, Ricci tensor and scalar curvature of the space-form respectively and Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$. We know (see [1]) that the ϕ-sectional curvature of a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$.

In a Riemannian manifold of dimension $(2n+1)$ the Weyl conformal curvature tensor is given by

$$\begin{aligned} C(X, Y)Z = R(X, Y)Z - \frac{1}{(2n-1)}\big[&S(Y, Z)X - S(X, Z)Y \\ &+ g(Y, Z)QX - g(X, Z)QY\big] \\ + \frac{r}{2n(2n-1)}\big[&g(Y, Z)X - g(X, Z)Y\big], \end{aligned} \tag{2.14}$$

for any X, Y and $Z \in T(M)$. A generalized Sasakian-space-form of dimension greater than three is said to be conformally flat if its Weyl conformal curvature tensor vanishes. It is known (see [12]) that a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is conformally flat if and only if $f_2 = 0$.

3. ξ-conformally flat generalized Sasakian-space-forms

In this section we study ξ-conformally flat generalized Sasakian-space-form. Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-forms. Putting $Z = \xi$ in (2.15), we obtain

$$\begin{aligned} C(X,Y)\xi = R(X,Y)\xi - \frac{1}{(2n-1)}\big[S(Y,\xi)X - S(X,\xi)Y \\ + \eta(Y)QX - \eta(X)QY\big] \\ + \frac{r}{2n(2n+1)}\big[\eta(Y)X - \eta(X)Y\big]. \end{aligned} \tag{3.1}$$

Using (2.7), (2.8) and (2.10) in (3.1), yields

$$\begin{aligned} C(X,Y)\xi = \Big[(f_1 - f_3) + \frac{r}{2n(2n-1)} \\ - \frac{4nf_1 + 3f_2 - (2n+1)f_3}{(2n+1)}\Big]\big[\eta(Y)X - \eta(X)Y\big]. \end{aligned} \tag{3.2}$$

Putting the value of r from (2.13) in (3.2), we get

$$C(X,Y)\xi = 0. \tag{3.3}$$

Therefore we conclude that the generalized Sasakian-space-form is ξ-conformally flat. Thus we can state the following:

Theorem 3.1. *A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is always ξ-conformally flat.*

4. ϕ-conformally flat generalized Sasakian-space-forms

In this section we consider ϕ-conformally flat generalized Sasakian-space-forms. Let the generalized Sasakian-space-form $M(f_1, f_2, f_3)$ be ϕ-conformally flat, i.e.,

$$g\big(C(\phi X, \phi Y)\phi Z, \phi W\big) = 0. \tag{4.1}$$

Now,

$$\begin{aligned}
g\big(C(\phi X,\phi Y)\phi Z,\phi W\big) = g\big(R(\phi X,\phi Y)\phi Z,\phi W\big)\\
-\frac{1}{2n-1}\big[S(\phi Y,\phi Z)g(\phi X,\phi W)\\
-S(\phi X,\phi Z)g(\phi Y,\phi W)\\
+g(\phi Y,\phi Z)S(\phi X,\phi W)\\
-S(\phi Y,\phi W)g(\phi X,\phi Z)\big]\\
+\frac{r}{2n(2n-1)}\big[g(\phi Y,\phi Z)g(\phi X,\phi W)\\
-g(\phi Y,\phi W)g(\phi X,\phi Z)\big].
\end{aligned}\tag{4.2}$$

Using (1.1), (2.6) and (2.13) in (4.2), we obtain

$$\begin{aligned}
g\big(C(\phi X,\phi Y)\phi Z,\phi W\big) = f_2\Big[\big\{2g(\phi X,Y)g(Z,\phi W)\\
+g(\phi X,Z)g(Y,\phi W)\\
-g(\phi Y,Z)g(X,\phi W)\big\}\\
-\frac{3}{(2n-1)}\big\{g(\phi Y,\phi Z)g(\phi X,\phi W)\\
-g(\phi X,\phi Z)g(\phi Y,\phi W)\big\}\Big].
\end{aligned}\tag{4.3}$$

Therefore from (4.1) and (4.3) we obtain either $f_2=0$, or

$$\begin{aligned}
&\big\{2g(\phi X,Y)g(Z,\phi W)+g(\phi X,Z)g(Y,\phi W)-g(\phi Y,Z)g(X,\phi W)\big\}\\
&-\frac{3}{(2n-1)}\big\{g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)\big\}=0.
\end{aligned}\tag{4.4}$$

Putting $W=\phi W$ in (4.5) and using (2.1), we obtain

$$\begin{aligned}
&\big\{-2g(\phi X,Y)g(Z,W)-g(\phi X,Z)g(Y,W)+g(\phi Y,Z)g(X,W)\big\}\\
&-\frac{3}{(2n-1)}\big\{-g(\phi Y,\phi Z)g(\phi X,W)+g(\phi X,\phi Z)g(\phi Y,W)\big\}=0.
\end{aligned}\tag{4.5}$$

Putting $Z=\xi$ in (4.5), yields

$$g(\phi X,Y)\eta(W)=0,\tag{4.6}$$

which implies either $\eta=0$ or $g(\phi X,Y)=0$. Both the cases do not occur. Therefore ϕ-conformally flat generalized Sasakian-space-form implies $f_2=0$.

Conversely, if $f_2 = 0$, then from (4.3) we obtain $g(C(\phi X, \phi Y)\phi Z), \phi W) = 0$ for all X, Y, Z and $W \in T(M)$. Thus $f_2 = 0$ implies the generalized Sasakian-space-form is ϕ-conformally flat.

Therefore we can state the following:

THEOREM 4.1. *A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is ϕ-conformally flat if and only if $f_2 = 0$.*

In [12] U. K. Kim proved that for a (2n+1)-dimensional generalized Sasakian-space-form the following holds:

(i) If $n > 1$, then M is conformally flat if and only if $f_2 = 0$.

(ii) If M is conformally flat and ξ is Killing vector field, then M is locally symmetric and has constant ϕ-sectional curvature.

Since conformally flatness implies ϕ-conformally flatness, hence in the view of the first part of the above theorem we have the following:

COROLLARY 4.1. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is ϕ-conformally flat if and only if it is conformally flat.*

Again, in the view of the second part of the above theorem we have the following:

COROLLARY 4.2. *For a $(2n+1)$-dimensional $(n > 1)$ ϕ-conformally flat generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with ξ as a Killing vector field, the space-form is locally symmetric and has constant ϕ-sectional curvature.*

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